

# The no-ghost theorem for string theory in curved backgrounds with a flat timelike direction

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## ABSTRACT

It is well-known that the standard no-ghost theorem can be extended to the general  $c = 26$  CFT with  $d$ -dimensional Minkowski spacetime  $\mathcal{M}^{(1,d-1)}$  and a compact unitary CFT  $K$  of central charge  $c_K = 26 - d$ . The theorem has been established under the assumption  $d \geq 2$  so far. We prove the no-ghost theorem for  $d = 1$ , *i.e.*, when only the timelike direction is flat. This is done using the technique of Frenkel, Garland and Zuckerman.

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# 1 Introduction

As is well-known, string theory generally contains negative norm states (ghosts) from timelike oscillators. However, they do not appear as physical states. This is the famous *no-ghost theorem* [1]-[11].<sup>1</sup>

When the background spacetime is curved, things are not clear though. First of all, nonstationary situations are complicated even in field theory. There is no well-defined ground state with respect to a time translation and the “particle interpretation” becomes ambiguous. Moreover, even for static or stationary backgrounds, it is not currently possible to show the no-ghost theorem from the knowledge of the algebra alone. In fact, refs. [16] and ref. [17] found ghosts for strings on three-dimensional anti-de Sitter space ( $AdS_3$ ) and three-dimensional black holes, respectively. There are references which have proposed resolutions to the ghost problem for  $AdS_3$  [18, 19]. However, the issue is not settled yet and the proof for the other backgrounds is still lacking.

Viewing the situation, we would like to ask the converse question: how general can the no-ghost theorem definitely apply? This is the theme of this paper.

In order to answer to the question, let us look at the known proofs more carefully. There are two approaches. The first approach uses the “old covariant quantization” (OCQ). Some proofs in this approach use the “DDF operators” to explicitly construct the observable Hilbert space to show the theorem [1, 12]. This proof assumes flat spacetime and cannot be extended to the more general cases. Goddard and Thorn’s proof [2, 6] is similar to this traditional one, but is formulated without explicit reference to the DDF operators. The proof is applicable when  $d \geq 2$  because it requires a flat light-cone vector.

There is another approach using the BRST quantization. These work more generally. Many proofs can be extended easily to the general  $c = 26$  CFT with  $d$ -dimensional Minkowski spacetime  $\mathcal{M}^{(1,d-1)}$  and a compact unitary CFT  $K$  of central charge  $c_K = 26 - d$  [9, 10, 11, 20]. However, all known proofs assume  $d \geq 2$  again.

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<sup>1</sup>For the NSR string, see refs. [12]-[15].

Therefore, the theorem has not been studied in detail when  $0 < d < 2$ . The purpose of this paper is to fill in the gap and to extend the proof to  $d = 1$ .

We show the no-ghost theorem using the technique of Frenkel, Garland and Zuckerman (FGZ) [9]. Their proof is different from the others. For example, the standard BRST quantization assumes  $d \geq 2$  in order to prove the “vanishing theorem,” *i.e.*, the BRST cohomology is trivial except at the zero ghost number. However, FGZ’s proof of the vanishing theorem essentially does not require this as we will see later. Establishing the no-ghost theorem for  $d = 1$  is then straightforward calculating the “index” and “signature” of the cohomology groups.

Thus, in a sense our result is obvious a priori, from refs. [9, 14, 15]. But it does not seem to be known well, so it is worth working out this point explicitly in detail.

The organization of the present paper is as follows. First, in the next section, we will set up our notations and review the BRST quantization of string theory briefly. In Section 3, we will see a standard proof of the vanishing theorem and review why the standard no-ghost theorem cannot be extended to  $d < 2$ . Then in Section 4, we will prove the vanishing theorem due to FGZ, following refs. [9, 14, 15]. We will use this result in Section 5 to prove the no-ghost theorem for  $d = 1$ .

For an earlier attempt of the  $d = 1$  proof, see ref. [11]. Thorn [4] used OCQ and proved the no-ghost theorem for  $1 \leq d \leq 25$ . The proof is based on OCQ, so the compact CFT is not assumed. However, there is no known way to give such a string consistent interactions at loop levels [21].

## 2 BRST Quantization

In this Section, we briefly review the BRST quantization of string theory [3, 20, 22]. We make the following assumptions:

- (i). Our world-sheet theory consists of  $d$  free bosons  $X^\mu$  ( $\mu = 0, \dots, d-1$ ) with signature  $(1, d-1)$  and a compact unitary CFT  $K$  of central charge  $c_K = 26 - d$ . We will focus on the  $d = 1$  case below, but the extension to  $d \geq 1$  is straightforward (Section 6).

- (ii). We assume that  $K$  is unitary and its spectrum is discrete and bounded below. Thus, all states in  $K$  lie in highest weight representations. The weight of highest weight states should have  $h^K \geq 0$  from the Kac determinant; therefore, the eigenvalue of  $L_0^K$  is always non-negative.
- (iii). The momentum of states is  $k^\mu \neq 0$ . (See Section 6 for the exceptional case  $k^\mu = 0$ .)

Then, the total  $L_m$  of the theory is given by  $L_m = L_m^X + L_m^g + L_m^K$ , where

$$L_m^X = \frac{1}{2} \sum_{n=-\infty}^{\infty} : \alpha_{m-n}^\mu \alpha_{\mu,n} :, \quad (1)$$

$$L_m^g = \sum_{n=-\infty}^{\infty} (m-n) : b_{m+n} c_{-n} : - \delta_m. \quad (2)$$

Here,

$$[\alpha_m^\mu, \alpha_n^\nu] = m \delta_{m+n} \eta^{\mu\nu}, \quad \{b_m, c_n\} = \delta_{m+n}, \quad (3)$$

and  $\delta_m = \delta_{m,0}$ . With the  $d$ -dimensional momentum  $k^\mu$ ,  $\alpha_0^\mu = \sqrt{2\alpha'} k^\mu$ .

The ghost number operator  $\hat{N}^g$  counts the number of  $c$  minus the number of  $b$  excitations: <sup>2</sup>

$$\hat{N}^g = \sum_{m=1}^{\infty} (c_{-m} b_m - b_{-m} c_m) = \sum_{m=1}^{\infty} (N_m^c - N_m^b). \quad (4)$$

We will call the total Hilbert space  $\mathcal{H}_{total}$ . Recall that the physical state conditions are

$$Q|\text{phys}\rangle = 0, \quad b_0|\text{phys}\rangle = 0. \quad (5)$$

These conditions imply

$$0 = \{Q, b_0\}|\text{phys}\rangle = L_0|\text{phys}\rangle. \quad (6)$$

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<sup>2</sup>The ghost zero modes will not matter to our discussion.  $\hat{N}^g$  is related to the standard ghost number operator  $N^g$  as  $N^g = \hat{N}^g + c_0 b_0 - \frac{1}{2}$ . Note that the operator  $\hat{N}^g$  is also normalized so that  $\hat{N}^g|\downarrow\rangle = 0$ .

Thus, we define the following subspaces of  $\mathcal{H}_{total}$ :

$$\mathcal{H} = \{\phi \in \mathcal{H}_{total} : b_0\phi = 0\}, \quad (7a)$$

$$\hat{\mathcal{H}} = \mathcal{H}^{L_0} = \{\phi \in \mathcal{H}_{total} : b_0\phi = L_0\phi = 0\}. \quad (7b)$$

We will consider the cohomology on  $\hat{\mathcal{H}}$  since  $Q$  takes  $\hat{\mathcal{H}}$  into itself from  $\{Q, b_0\} = L_0$  and  $[Q, L_0] = 0$ . The subspace  $\mathcal{H}$  will be useful in our proof of the vanishing theorem (Section 4).

The Hilbert space  $\hat{\mathcal{H}}$  is classified according to mass eigenvalues.  $\hat{\mathcal{H}}$  at a particular mass level will be often written as  $\hat{\mathcal{H}}(k^2)$ . For a state  $|\phi\rangle \in \hat{\mathcal{H}}(k^2)$ ,

$$L_0|\phi\rangle = (\alpha'k^2 + L_0^{int})|\phi\rangle = 0, \quad (8)$$

where  $L_0^{int}$  counts the level number. One can further take an eigenstate of  $\hat{N}^g$  since  $[L_0^{int}, \hat{N}^g] = 0$ .  $\hat{\mathcal{H}}$  is decomposed by the eigenvalues of  $\hat{N}^g$  as

$$\hat{\mathcal{H}} = \bigoplus_{n \in \mathbf{Z}} \hat{\mathcal{H}}^n. \quad (9)$$

We define the raising operators as  $\alpha_{-m}^\mu, b_{-m}, c_{-m}, x^\mu$  and  $c_0$ . The ground state in  $\hat{\mathcal{H}}(k^2)$  is given by

$$|0; k\rangle \otimes |h^K\rangle = e^{ik \cdot x} |0; \downarrow\rangle \otimes |h^K\rangle, \quad (10)$$

where  $|0; \downarrow\rangle$  is the vacuum state annihilated by all lowering operators and  $|h^K\rangle$  is a highest weight state in  $K$ . Then,  $\hat{\mathcal{H}}(k^2)$  is written as

$$\hat{\mathcal{H}}(k^2) = (\mathcal{F}(\alpha_{-m}^\mu, b_{-m}, c_{-m}; k) \otimes \mathcal{H}_K)^{L_0}. \quad (11)$$

Here,  $*^{L_0}$  denotes the  $L_0$ -invariant subspace:  $F^{L_0} = F \cap \text{Ker } L_0$ . A state in  $\mathcal{H}_K$  is constructed by Verma modules of  $K$ . The Fock space  $\mathcal{F}(\alpha_{-m}^\mu, b_{-m}, c_{-m}; k)$  is spanned by all states of the form

$$|N; k\rangle = \prod_{\mu=0}^{d-1} \prod_{m=1}^{\infty} \frac{(\alpha_{-m}^\mu)^{N_m^\mu}}{\sqrt{m^{N_m^\mu} N_m^\mu!}} \prod_{m=1}^{\infty} (b_{-m})^{N_m^b} \prod_{m=1}^{\infty} (c_{-m})^{N_m^c} |0; k\rangle, \quad (12)$$

where  $N_m^\mu = \frac{1}{m} \alpha_{-m}^\mu \alpha_{\mu, m}$  are the number operators for  $\alpha_{-m}^\mu$ . In terms of the number operators,

$$L_0^{int} = \sum_{m=1}^{\infty} m \left( N_m^b + N_m^c + \sum_{\mu=0}^{d-1} N_m^\mu \right) + L_0^K - 1. \quad (13)$$

The BRST operator

$$Q = \sum_{m=-\infty}^{\infty} (L_{-m}^X + L_{-m}^K) c_m - \frac{1}{2} \sum_{m,n=-\infty}^{\infty} (m-n) : c_{-m} c_{-n} b_{m+n} : - c_0 \quad (14)$$

can be decomposed in terms of ghost zero modes as follows:

$$Q = \hat{Q} + c_0 L_0 + b_0 M, \quad (15)$$

where  $M = -2 \sum_{m=1}^{\infty} m c_{-m} c_m$  and  $\hat{Q}$  is the collection of the terms in  $Q$  without  $b_0$  or  $c_0$ . Using the above decomposition (15), for a state  $|\phi\rangle \in \hat{\mathcal{H}}$ ,

$$Q|\phi\rangle = \hat{Q}|\phi\rangle. \quad (16)$$

Therefore, the physical state condition reduces to

$$\hat{Q}|\phi\rangle = 0. \quad (17)$$

Also,  $\hat{Q}^2 = 0$  on  $\hat{\mathcal{H}}$  from Eq. (16). Thus,  $\hat{Q} : \hat{\mathcal{H}}^n \rightarrow \hat{\mathcal{H}}^{n+1}$  defines a BRST complex, which is called the *relative* BRST complex. So, we can define  $\hat{\mathcal{H}}_c, \hat{\mathcal{H}}_e \subset \hat{\mathcal{H}}$  by

$$\hat{Q}\hat{\mathcal{H}}_c = 0, \quad \hat{\mathcal{H}}_e = \hat{Q}\hat{\mathcal{H}}, \quad (18)$$

and define the relative BRST cohomology of  $Q$

$$\hat{\mathcal{H}}_{obs} = \hat{\mathcal{H}}_c / \hat{\mathcal{H}}_e. \quad (19)$$

In terms of the cohomology group,  $\hat{\mathcal{H}}_{obs}(k^2) = \oplus_{n \in \mathbf{Z}} H^n(\hat{Q}(k), \hat{\mathcal{H}}(k^2))$ .

Now, the inner product in  $\mathcal{H}_{total}$  is given by

$$\langle 0, I; k |_{c_0} | 0, I'; k' \rangle = (2\pi)^d \delta^d(k - k') \delta_{I, I'}, \quad (20)$$

where  $I$  labels the states of the compact CFT  $K$ . We take the basis  $I$  to be orthonormal. Then, the inner product  $\langle || \rangle$  in  $\hat{\mathcal{H}}$  is defined by  $\langle || \rangle$  as follows:

$$\langle 0, I; k |_{c_0} | 0, I'; k' \rangle = 2\pi \delta(k^2 - k'^2) \langle 0, I; k || 0, I'; k' \rangle. \quad (21)$$

The inner products of the other states follow from the algebra with the hermiticity property  $b_m^\dagger = b_{-m}$ ,  $c_m^\dagger = c_{-m}$  and  $(\alpha_m^\mu)^\dagger = \alpha_{-m}^\mu$ .<sup>3</sup>

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<sup>3</sup>We will write  $\langle \cdots || \cdots \rangle$  as  $\langle \cdots | \cdots \rangle$  below.

### 3 The Vanishing Theorem and Standard Proofs

In order to prove the no-ghost theorem, it is useful to show the following theorem:

**Theorem 3.1 (The Vanishing Theorem).** *The  $\hat{Q}$ -cohomology can be nonzero only at  $\hat{N}^g = 0$ , i.e.,  $H^n(\hat{Q}, \hat{\mathcal{H}}) = 0$  for  $n \neq 0$ .*

To prove this, we use the notion of *filtration*. We first explain the method and then give an example of filtration used in [3, 22]. The filtration is part of the reason why  $d \geq 2$  in standard proofs.

A filtration is a procedure to break up  $\hat{Q}$  according to a quantum number  $N_f$  (filtration degree):

$$\hat{Q} = Q_0 + Q_1 + \cdots + Q_N, \quad (22)$$

where

$$[N_f, Q_m] = mQ_m. \quad (23)$$

We also require

$$[N_f, \hat{N}^g] = [N_f, L_0] = 0. \quad (24)$$

Then,  $\hat{\mathcal{H}}$  breaks up according to the filtration degree  $N_f (= q)$  as well as the ghost number  $\hat{N}^g (= n)$ :

$$\hat{\mathcal{H}} = \bigoplus_{q,n \in \mathbf{Z}} \hat{\mathcal{H}}^{n;q}. \quad (25)$$

If  $\hat{\mathcal{H}}^q$  can be nonzero only for a finite range of degrees, the filtration is called *bounded*.

The nilpotency of  $\hat{Q}^2$  implies

$$\sum_{\substack{m,n \\ m+n=l}} Q_m Q_n = 0, \quad l = 0, \dots, 2N \quad (26)$$

since they have different values of  $N_f$ . In particular,

$$Q_0^2 = 0. \quad (27)$$

The point is that we can first study the cohomology of  $Q_0 : \hat{\mathcal{H}}^{n;q} \rightarrow \hat{\mathcal{H}}^{n+1;q}$ . This is easier since  $Q_0$  is often simpler than  $\hat{Q}$ . Knowing the cohomology of  $Q_0$  then tells us about the cohomology of  $\hat{Q}$ . In fact, one can show the following lemma:

**Lemma 3.1.** *If the  $Q_0$ -cohomology is trivial, so is the  $\hat{Q}$ -cohomology.*

*Proof.* <sup>4</sup> Let  $\phi$  be a state of ghost number  $\hat{N}^g = n$  and  $\hat{Q}$ -invariant ( $\phi \in \hat{\mathcal{H}}_c^n$ ). Assuming that the filtration is bounded, we write

$$\phi = \phi_k + \phi_{k+1} + \cdots + \phi_p, \quad (28)$$

where  $\phi_q \in \hat{\mathcal{H}}^{n;q}$ . Then,

$$\hat{Q}\phi = (Q_0\phi_k) + (Q_0\phi_{k+1} + Q_1\phi_k) + \cdots + (Q_N\phi_p). \quad (29)$$

Each parenthesis vanishes separately since they carry different  $N_f$ . So,  $Q_0\phi_k = 0$ . The  $Q_0$ -cohomology is trivial by assumption, thus  $\phi_k = Q_0\chi_k$ . But then  $\phi - \hat{Q}\chi_k$ , which is cohomologous to  $\phi$ , has no  $N_f = k$  piece. By induction, we can eliminate all  $\phi_q$ , so  $\phi = \hat{Q}(\chi_k + \cdots + \chi_p)$ ;  $\phi$  is actually  $\hat{Q}$ -exact.  $\square$

Moreover, one can show that the  $Q_0$ -cohomology is isomorphic to that of  $\hat{Q}$  if the  $Q_0$ -cohomology is nontrivial for at most one filtration degree [20, 22]. We will not present the proof because our derivation does not need this. In the language of a spectral sequence [24], the first term and the limit term of the sequence are

$$E_1 \cong \bigoplus_q H(\hat{\mathcal{H}}^q, d_0), \quad E_\infty \cong H(\hat{\mathcal{H}}, \hat{Q}). \quad (30)$$

The above results state that the sequence collapses after the first term:

$$E_1 \cong E_\infty. \quad (31)$$

Then, a standard proof proceeds to show that states in the nontrivial degree are in fact light-cone spectra, and thus there is no ghost in the  $\hat{Q}$ -cohomology [20].

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<sup>4</sup>The following argument is due to ref. [22].



Now, we have to find an appropriate filtration and show that the  $Q_0$ -cohomology is trivial if  $\hat{N}^g \neq 0$ . This will complete the proof of the vanishing theorem. The standard proof of the theorem uses the following filtration [3, 22]:<sup>5</sup>

$$N_f^{(KO)} = \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1}{m} \alpha_{-m}^- \alpha_m^+ + \hat{N}^g. \quad (32)$$

The degree  $N_f^{(KO)}$  counts the number of  $\alpha^+$  minus the number of  $\alpha^-$  excitations. So, this filtration assumes two flat directions. The degree zero part of  $\hat{Q}$  is

$$Q_0^{(KO)} = -\sqrt{2\alpha'} k^+ \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} c_m \alpha_{-m}^-. \quad (33)$$

The operator  $Q_0^{(KO)}$  is nilpotent since  $\alpha_m^-$  commute and  $c_m$  anticommute. Obviously, we cannot use  $\alpha_m^0$  in place of  $\alpha_m^-$  since  $\alpha_m^0$  do not commute. Thus, we have to take a different approach for  $d = 1$ .

Goddard and Thorn's proof of the no-ghost theorem [2, 6] uses a different technique, but this proof also assumes two flat directions; the proof makes use of a light-cone vector.

## 4 The Vanishing Theorem (FGZ)

Since we want to show the no-ghost theorem for  $d = 1$ , we cannot use  $N_f^{(KO)}$  as our filtration degree. Fortunately, there is a different proof of the vanishing theorem [9, 14, 15], which uses a different filtration. Their filtration is unique in that  $Q_0$  can be actually written as a sum of two differentials  $d'$  and  $d''$ . This effectively reduces the problem to a “ $c = 1$ ” CFT, which contains the timelike part and the  $b$  ghost part. Then, a Künneth formula relates the theorem to the whole complex. This is the reason why the proof does not

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<sup>5</sup>The  $\hat{N}^g$  piece is not really necessary. We include this to make the filtration degree non-negative.

require  $d \geq 2$ . In addition, in this approach the matter Virasoro generators themselves play a role similar to that of the light-cone oscillators in Kato-Ogawa's approach. In this section, we will prove the theorem using the technique of refs. [9, 14, 15], but for more mathematically rigorous discussion, consult the original references.

*Proof of the vanishing theorem for  $d = 1$ .* FGZ's filtration is originally given for the  $d = 26$  flat spacetime as

$$N_f^{(FGZ)} = -L_0^{X(d=26)} + \sum_{m=1}^{\infty} m(N_m^c - N_m^b). \quad (34)$$

The filtration itself does not require  $d \geq 2$ ; this filtration can be naturally used for  $d = 1$  replacing  $L_0^{X(d=26)}$  with  $L_0^X$ . Then, the modified filtration assigns the following degrees to the operators:

$$\text{fdeg}(c_m) = |m|, \quad \text{fdeg}(b_m) = -|m|, \quad (35a)$$

$$\text{fdeg}(L_m^X) = m, \quad \text{fdeg}(L_m^K) = 0. \quad (35b)$$

The operator  $N_f^{(FGZ)}$  satisfies conditions (24) and the degree of each term in  $\hat{Q}$  is non-negative. Because the eigenvalue of  $L_0^{int}$  is bounded below from Eq. (13), the total number of oscillators for a given mass level is bounded. Thus, the degree for the states is bounded for each mass level. Note that the unitarity of the compact CFT  $K$  is essential for the filtration to be bounded.

The degree zero part of  $\hat{Q}$  is given by

$$Q_0^{(FGZ)} = d' + d'', \quad (36a)$$

$$d' = \sum_{m>0} c_m L_{-m}^X + \sum_{m,n>0} \frac{1}{2} (m-n) b_{-m-n} c_m c_n, \quad (36b)$$

$$d'' = - \sum_{m,n>0} \frac{1}{2} (m-n) c_{-m} c_{-n} b_{m+n}. \quad (36c)$$

We break  $\hat{\mathcal{H}}$  as follows:

$$\hat{\mathcal{H}} = (\mathcal{F}(\alpha_{-m}^0, b_{-m}, c_{-m}; k^0) \otimes \mathcal{H}_K)^{L_0} \quad (37a)$$

$$= (\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0) \otimes \mathcal{F}(c_{-m}) \otimes \mathcal{H}_K)^{L_0}. \quad (37b)$$

The Hilbert spaces  $\hat{\mathcal{H}}$ ,  $\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)$  and  $\mathcal{F}(c_{-m})$  are decomposed according to the ghost number  $\hat{N}^g = n$ :

$$\hat{\mathcal{H}}^n = \left( \left( \bigoplus_{\substack{n=N^c-N^b \\ N^c, N^b \geq 0}} \mathcal{F}^{-N^b}(\alpha_{-m}^0, b_{-m}; k^0) \otimes \mathcal{F}^{N^c}(c_{-m}) \right) \otimes \mathcal{H}_K \right)^{L_0}. \quad (38)$$

From Eqs. (36), the differentials act as follows:

$$Q_0^{(FGZ)} : \mathcal{H}^n \rightarrow \mathcal{H}^{n+1}, \quad (39a)$$

$$d' : \mathcal{F}^n(\alpha_{-m}^0, b_{-m}; k^0) \rightarrow \mathcal{F}^{n+1}(\alpha_{-m}^0, b_{-m}; k^0), \quad (39b)$$

$$d'' : \mathcal{F}^n(c_{-m}) \rightarrow \mathcal{F}^{n+1}(c_{-m}), \quad (39c)$$

and  $d'^2 = d''^2 = 0$ . Thus,  $\mathcal{F}^n(\alpha_{-m}^0, b_{-m}; k^0)$  and  $\mathcal{F}^n(c_{-m})$  are complexes with differentials  $d'$  and  $d''$ . Note that  $Q_0^{(FGZ)}$  is the differential for  $\mathcal{H}^n$  as well as for  $\hat{\mathcal{H}}^n$ .

Then, the Künneth formula (Appendix A) relates the cohomology group of  $\mathcal{H}$  to the ones of  $\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)$  and  $\mathcal{F}(c_{-m})$ :

$$H^n(\mathcal{H}) = \left( \bigoplus_{\substack{n=N^c-N^b \\ N^c, N^b \geq 0}} H^{-N^b}(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)) \otimes H^{N^c}(\mathcal{F}(c_{-m})) \right) \otimes \mathcal{H}_K. \quad (40)$$

Later we will prove the following lemma:

**Lemma 4.1.**  $H^{-N^b}(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)) = 0$  if  $N^b > 0$  and  $(k^0)^2 > 0$ .

Then, Eq. (40) reduces to

$$H^n(\mathcal{H})^{L_0} = \left( \bigoplus_{n=N^c} H^0(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)) \otimes H^{N^c}(\mathcal{F}(c_{-m})) \otimes \mathcal{H}_K \right)^{L_0}, \quad (41)$$

which leads to  $H^n(\mathcal{H})^{L_0} = 0$  for  $n < 0$  because  $N^c \geq 0$ . The cohomology group  $H^n(\mathcal{H})^{L_0}$  is not exactly what we want. However, Lian and Zuckerman have shown that

$$H^n(\hat{\mathcal{H}}) \cong H^n(\mathcal{H})^{L_0}. \quad (42)$$

See pages 325-326 of ref. [14]. Thus,

$$H^n(\hat{\mathcal{H}}) = 0 \quad \text{if } n < 0. \quad (43)$$

In Section 5, we will prove the Poincaré duality theorem (Theorem 5.2):  $H^n(\hat{\mathcal{H}}) = H^{-n}(\hat{\mathcal{H}})$ . Therefore,

$$H^n(\hat{\mathcal{H}}) = 0 \quad \text{if } n \neq 0. \quad (44)$$

This is the vanishing theorem for  $d = 1$ .  $\square$

Now we will show Lemma 4.1. The proof is twofold: the first is to map the  $c = 1$  Fock space  $\mathcal{F}(\alpha_{-m}^0; k^0)$  to a Verma module, and the second is to show the lemma using the Verma module.

Let  $\mathcal{V}(c, h)$  be a Verma module with highest weight  $h$  and central charge  $c$ . Then, we will first show the isomorphism

$$\mathcal{F}(\alpha_{-m}^0; k^0) \cong \mathcal{V}(1, h^X) \quad \text{if } (k^0)^2 > 0. \quad (45)$$

Here,  $h^X = -\alpha'(k^0)^2$ . This is plausible from the defining formula of  $L_m^X$ ,

$$L_{-m}^X = \sqrt{2\alpha'} k_0 \alpha_{-m}^0 + \cdots, \quad (46)$$

where  $+\cdots$  denotes terms with more than one oscillators. The actual proof is rather similar to an argument in [4, 23].

*Proof of Eq. (45).* The number of states of the Fock space  $\mathcal{F}(\alpha_{-m}^0; k^0)$  and that of the Verma module  $\mathcal{V}(1, h^X)$  are the same for a given level  $N$ . Thus, the Verma module furnishes a basis of the Fock space if the members of a conformal family,

$$|h^X, \{\lambda\}\rangle = L_{-\lambda_1}^X L_{-\lambda_2}^X \cdots L_{-\lambda_M}^X |h^X\rangle, \quad (47)$$

are linearly independent, where  $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_M$ . This can be shown using the Kac determinant.

Consider the matrix of inner products for the states at level  $N$ :

$$\mathcal{M}_{\{\lambda\}, \{\lambda'\}}^N(c, h^X) = \langle h^X, \{\lambda\} | h^X, \{\lambda'\} \rangle, \quad \sum_i \lambda_i = N. \quad (48)$$

The Kac determinant is then given by

$$\det[\mathcal{M}^N(c, h^X)] = K_N \prod_{1 \leq rs \leq N} (h^X - h_{r,s})^{P(N-rs)}, \quad (49)$$

where  $K_N$  is a positive constant and the multiplicity of the roots,  $P(N-rs)$ , is the partition of  $N-rs$ . The zeros of the Kac determinant are at

$$h_{r,s} = \frac{c-1}{24} + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2, \quad (50)$$

where

$$\alpha_{\pm} = \frac{1}{\sqrt{24}}(\sqrt{1-c} \pm \sqrt{25-c}). \quad (51)$$

For  $c=1$ ,  $\alpha_{\pm} = \pm 1$  so that  $h_{r,s} = (r-s)^2/4 \geq 0$ . Thus, the states  $|h^X, \{\lambda\}\rangle$  are linearly independent if  $h^X < 0$ .  $\square$

Let us check what spectrum actually appears in  $\hat{\mathcal{H}}$ . Using assumption (ii) of Section 2, Eqs. (8) and (13), we get  $h^X \leq 1$  for a state in  $\hat{\mathcal{H}}$ . Also,  $h^X \neq 0$  from assumption (iii).<sup>6</sup> Thus, we need to consider the Fock spaces  $\mathcal{F}(\alpha_{-m}^0; k^0)$  with  $h^X \leq 1$  ( $h^X \neq 0$ ). The ones with  $h^X < 0$  are expressed by Verma modules from Eq. (45). On the other hand, the ones with  $0 < h^X \leq 1$  are not. However, there is only the ground state in this region as the states in  $\hat{\mathcal{H}}$ . This state has  $\hat{N}^g = 0$ , so the state does not affect the vanishing theorem.

The isomorphism (45) is essential for proving the vanishing theorem. In the language of FGZ, what we have shown is that  $\mathcal{F}(\alpha_{-m}^0; k^0)$  is an “ $\mathcal{L}_-$ -free module,” which is a prime assumption of the vanishing theorem (Theorem 1.12 of [9]). The proof of Lemma 4.1 is now straightforward using Eq. (45) and an argument given in [15]:

*Proof of Lemma 4.1.* Using Eq. (45), a state  $|\phi\rangle \in \mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)$  can be written as

$$|\phi\rangle = b_{-i_1} \dots b_{-i_L} L_{-\lambda_1}^X \dots L_{-\lambda_M}^X |h^X\rangle, \quad (52)$$

---

<sup>6</sup>The Verma module  $\mathcal{V}(1, 0)$  fails to furnish the basis of  $\mathcal{F}(\alpha_{-m}^0; 0)$  at the first level because  $L_{-1}^X |h^X = 0\rangle = 0$  for  $d=1$ .

where  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_M$  and  $0 < i_1 < i_2 < \dots < i_L$ . Note that the states in  $\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)$  all have nonpositive ghost number:  $\hat{N}^g|\phi\rangle = -L|\phi\rangle$ .

We define a new filtration degree  $N_f^{(FK)}$  as

$$N_f^{(FK)}|\phi\rangle = -(L + M)|\phi\rangle \quad (53)$$

which corresponds to

$$\text{fdeg}(L_{-m}^X) = \text{fdeg}(b_{-m}) = -1 \quad \text{for } m > 0. \quad (54)$$

The algebra then determines  $\text{fdeg}(c_m) = 1$  (for  $m > 0$ ) from the assignment. The operator  $N_f^{(FK)}$  satisfies conditions (24) and the degree of each term in  $d'$  is non-negative. The degree zero part of  $d'$  is given by

$$d'_0 = \sum_{m>0} c_m L_{-m}^X. \quad (55)$$

Since we want a bounded filtration, break up  $\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)$  according to  $L_0$  eigenvalue  $l_0$  :

$$\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0) = \bigoplus_{l_0} \mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)^{l_0}, \quad (56)$$

where

$$\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)^{l_0} = \mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0) \cap \text{Ker}(L_0 - l_0). \quad (57)$$

Note that  $\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)^{l_0}$  is finite dimensional since  $|\phi\rangle \in \mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)^{l_0}$  satisfies

$$\sum_{k=1}^L i_k + \sum_{k=1}^M \lambda_k + h^X = l_0. \quad (58)$$

Thus, the above filtration is bounded for each  $\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)^{l_0}$ .

We first consider the  $d'_0$ -cohomology on  $\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)^{l_0}$  for each  $l_0$ . Define an operator  $\Gamma$  such as

$$\Gamma|\phi\rangle = \sum_{l=1}^M b_{-\lambda_l} (b_{-i_1} \dots b_{-i_L}) L_{-\lambda_1}^X \dots \widehat{L_{-\lambda_l}^X} \dots L_{-\lambda_M}^X |h^X\rangle, \quad (59)$$

where  $\widehat{L_{-\lambda_i}^X}$  means that the term is missing (When  $M = 0$ ,  $\Gamma|\phi\rangle \stackrel{\text{def}}{=} 0$ ). Then, it is straightforward to show that

$$\{d'_0, \Gamma\}|\phi\rangle = (L + M)|\phi\rangle. \quad (60)$$

The operator  $\Gamma$  is called a *homotopy operator* for  $d'_0$ . Its significance is that the  $d'_0$ -cohomology is trivial except for  $L + M = 0$ . If  $|\phi\rangle$  is closed, then

$$|\phi\rangle = \frac{\{d'_0, \Gamma\}}{L + M}|\phi\rangle = \frac{1}{L + M}d'_0\Gamma|\phi\rangle. \quad (61)$$

Thus, a closed state  $|\phi\rangle$  is actually an exact state if  $L + M \neq 0$ . Therefore, the  $d'_0$ -cohomology is trivial if  $\hat{N}^g < 0$  since  $\hat{N}^g = -L$ . And now, again using Lemma 3.1, the  $d'$ -cohomology  $H^n(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)^{l_0})$  is trivial if  $n < 0$ .

Because  $[d', L_0] = 0$ , we can define

$$H^n(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0))^{l_0} = H^n(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)) \cap \text{Ker}(L_0 - l_0). \quad (62)$$

Furthermore, the isomorphism

$$H^n(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0))^{l_0} \cong H^n(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)^{l_0}) \quad (63)$$

can be established. Consequently,  $H^n(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)) = 0$  if  $n < 0$ .  $\square$

## 5 The No-Ghost Theorem

Having shown the vanishing theorem, it is straightforward to show the no-ghost theorem:

**Theorem 5.1 (The No-Ghost Theorem).**  $\hat{\mathcal{H}}_{obs}$  is a positive definite space when  $d \geq 1$ .

The calculation below is essentially the same as the one in refs. [9, 10, 15], but we repeat it here for completeness.

In order to prove the theorem, the notion of *signature* is useful. For a vector space  $V$  with an inner product, we can choose a basis  $e_a$  such that

$$\langle e_a | e_b \rangle = \delta_{ab} C_a, \quad (64)$$

where  $C_a \in \{0, \pm 1\}$ . Then, the signature of  $V$  is defined as

$$\text{sign}(V) = \sum_a C_a, \quad (65)$$

which is independent of the choice of  $e_a$ . Note that if  $\text{sign}(V) = \dim(V)$ , all the  $C_a$  are 1, so  $V$  has positive definite norm.

So, the statement of the no-ghost theorem is equivalent to <sup>7</sup>

$$\text{sign}(V_i^{obs}) = \dim(V_i^{obs}). \quad (66)$$

This can be replaced as a more useful form

$$\sum_i e^{-\lambda \alpha' m_i^2} \text{sign}(V_i^{obs}) = \sum_i e^{-\lambda \alpha' m_i^2} \dim(V_i^{obs}), \quad (67)$$

where  $\lambda$  is a constant. Equation (66) can be retrieved from Eq. (67) by expanding in powers of  $\lambda$ . We will write Eq. (67) as

$$\text{tr}_{obs} e^{-\lambda L_0^{int}} C = \text{tr}_{obs} e^{-\lambda L_0^{int}}, \quad (68)$$

where  $C$  is an operator which gives eigenvalues  $C_a$ .

Equation (68) is not easy to calculate; however, the following relation is straightforward to prove:

$$\text{tr} e^{-\lambda L_0^{int}} C = \text{tr} e^{-\lambda L_0^{int}} (-)^{\hat{N}^g}. \quad (69a)$$

Here, the trace is taken over  $V_i$  and we take a basis which diagonalizes  $(-)^{\hat{N}^g}$ . Thus, we can prove Eq. (68) by showing Eq. (69a) and

$$\text{tr} e^{-\lambda L_0^{int}} (-)^{\hat{N}^g} = \text{tr}_{obs} e^{-\lambda L_0^{int}}, \quad (69b)$$

$$\text{tr} e^{-\lambda L_0^{int}} C = \text{tr}_{obs} e^{-\lambda L_0^{int}} C. \quad (69c)$$

Thus, the trace weighted by  $(-)^{\hat{N}^g}$  is an *index*.

---

<sup>7</sup>In this section, we will also write  $V_i^{obs} = \hat{\mathcal{H}}_{obs}(k^2)$  and  $V_i = \hat{\mathcal{H}}(k^2)$ , where the subscript  $i$  labels different mass levels.



*Proof of Eq. (69b).* At each mass level, states  $\varphi_m$  in  $V_i$  are classified into two kinds of representations: BRST singlets  $\phi_{\bar{a}} \in V_i^{obs}$  and BRST doublets  $(\chi_a, \psi_a)$ , where  $\chi_a = \hat{Q}\psi_a$ . The ghost number of  $\chi_a$  is the ghost number of  $\psi_a$  plus 1. Therefore,  $(-)^{\hat{N}^g}$  causes these pairs of states to cancel in the index and only the singlets contribute:

$$\text{tr} e^{-\lambda L_0^{int}} (-)^{\hat{N}^g} = \text{tr}_{obs} e^{-\lambda L_0^{int}} (-)^{\hat{N}^g} \quad (70)$$

$$= \text{tr}_{obs} e^{-\lambda L_0^{int}}. \quad (71)$$

We have used the vanishing theorem on the last line.  $\square$

*Proof of Eq. (69c).* At a given mass level, the matrix of inner products among  $|\varphi_m\rangle$  takes the form

$$\langle \varphi_m | \varphi_n \rangle = \begin{pmatrix} \langle \chi_a | \\ \langle \psi_a | \\ \langle \phi_{\bar{a}} | \end{pmatrix} (|\chi_b\rangle, |\psi_b\rangle, |\phi_{\bar{b}}\rangle) = \begin{pmatrix} 0 & M & 0 \\ M^\dagger & A & B \\ 0 & B^\dagger & D \end{pmatrix}. \quad (72)$$

We have used  $\hat{Q}^\dagger = \hat{Q}$ ,  $\langle \chi | \chi \rangle = \langle \chi | \hat{Q} | \psi \rangle = 0$  and  $\langle \chi | \phi \rangle = \langle \psi | \hat{Q} | \phi \rangle = 0$ . If  $M$  were degenerate, there would be a state  $\chi_a$  which is orthogonal to all states in  $V_i$ . Thus, the matrix  $M$  should be nondegenerate. (Similarly, the matrix  $D$  should be nondegenerate as well.) So, a change of basis

$$\begin{aligned} |\chi'_a\rangle &= |\chi_a\rangle, \\ |\psi'_a\rangle &= |\psi_a\rangle - \frac{1}{2}(M^{-\dagger}A)_{ba}|\chi_b\rangle, \\ |\phi'_{\bar{a}}\rangle &= |\phi_{\bar{a}}\rangle - (M^{-\dagger}B)_{b\bar{a}}|\chi_b\rangle, \end{aligned} \quad (73)$$

sets  $A = B = 0$ . Finally, going to a basis,

$$\begin{aligned} |\chi''_a\rangle &= \frac{1}{\sqrt{2}}(|\chi'_a\rangle + M_{ba}^{-1}|\psi'_b\rangle), \\ |\psi''_a\rangle &= \frac{1}{\sqrt{2}}(|\chi'_a\rangle - M_{ba}^{-1}|\psi'_b\rangle), \\ |\phi''_{\bar{a}}\rangle &= |\phi'_{\bar{a}}\rangle, \end{aligned} \quad (74)$$

the inner product  $\langle \varphi''_m | \varphi''_n \rangle$  becomes block-diagonal:

$$\langle \varphi''_m | \varphi''_n \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & D \end{pmatrix}. \quad (75)$$

Therefore, BRST doublets again make no net contribution:

$$\text{tr } e^{-\lambda L_0^{int}} C = \text{tr}_{obs} e^{-\lambda L_0^{int}} C. \quad (76)$$

This proves Eq. (69c).  $\square$

One can indeed check that  $M$  and  $D$  are nondegenerate. The inner product in  $V_i$  is written as the product of inner products in  $\mathcal{F}(\alpha_{-m}^0; k^0)$ , ghost sector and  $\mathcal{H}_K$ . The inner product in  $\mathcal{F}(\alpha_{-m}^0; k^0)$  is easily seen to be diagonal and nondegenerate. For the ghost sector, the inner product becomes diagonal and nondegenerate as well by taking the basis  $p_m = (b_m + c_m)/\sqrt{2}$  and  $m_m = (b_m - c_m)/\sqrt{2}$ , where

$$\{p_m, p_n\} = \delta_{m+n}, \quad \{p_m, m_n\} = 0, \quad \{m_m, m_n\} = -\delta_{m+n}. \quad (77)$$

$\mathcal{H}_K$  is assumed to have a positive-definite inner product. Therefore, the matrix  $\langle \varphi_m | \varphi_n \rangle$  is nondegenerate. Consequently, the matrices  $M$  and  $D$  are also nondegenerate.

The inner product is nonvanishing only between the states with opposite ghost numbers. Since  $D$  is nondegenerate, BRST singlets of opposite ghost number must pair up. We have therefore established the Poincaré duality theorem as well:

**Theorem 5.2 (Poincaré Duality).**  $H^{\hat{N}^g}(\hat{\mathcal{H}}) = H^{-\hat{N}^g}(\hat{\mathcal{H}})$ .

*Proof of Eq. (69a).* We prove Eq. (69a) by explicitly calculating the both sides.

In order to calculate the left-hand side of Eq. (69a), take an orthonormal basis of definite  $N_m^p$ ,  $N_m^m$  [the basis (77)],  $N_m^0$  and an orthonormal basis of  $\mathcal{H}_K$ . Then,  $C = (-)^{N_m^m + N_m^0}$ . Similarly, for the right-hand side, take an orthonormal basis of definite  $N_m^b$ ,  $N_m^c$ ,  $N_m^0$  and an orthonormal basis of  $\mathcal{H}_K$ .

From these relations, the left-hand side of Eq. (69a) becomes

$$\begin{aligned}
& \text{tr } e^{-\lambda L_0^{int}} C \\
&= e^\lambda \prod_{m=1}^{\infty} \left( \sum_{N_m^p=0}^1 e^{-\lambda m N_m^p} \right) \left( \sum_{N_m^m=0}^1 e^{-\lambda m N_m^m} (-)^{N_m^m} \right) \\
&\quad \times \left( \sum_{N_m^0=0}^{\infty} e^{-\lambda m N_m^0} (-)^{N_m^0} \right) \text{tr}_{\mathcal{H}_K} e^{-\lambda L_0^K} \\
&= e^\lambda \prod_m (1 + e^{-\lambda m})(1 - e^{-\lambda m})(1 + e^{-\lambda m})^{-1} \text{tr}_{\mathcal{H}_K} e^{-\lambda L_0^K} \\
&= e^\lambda \prod_m (1 - e^{-\lambda m}) \text{tr}_{\mathcal{H}_K} e^{-\lambda L_0^K}. \tag{78}
\end{aligned}$$

The right-hand side becomes

$$\begin{aligned}
& \text{tr } e^{-\lambda L_0^{int}} (-)^{\hat{N}^g} \\
&= e^\lambda \prod_{m=1}^{\infty} \left( \sum_{N_m^b=0}^1 e^{-\lambda m N_m^b} (-)^{N_m^b} \right) \left( \sum_{N_m^c=0}^1 e^{-\lambda m N_m^c} (-)^{N_m^c} \right) \\
&\quad \times \left( \sum_{N_m^0=0}^{\infty} e^{-\lambda m N_m^0} \right) \text{tr}_{\mathcal{H}_K} e^{-\lambda L_0^K} \\
&= e^\lambda \prod_m (1 - e^{-\lambda m})(1 - e^{-\lambda m})(1 - e^{-\lambda m})^{-1} \text{tr}_{\mathcal{H}_K} e^{-\lambda L_0^K} \\
&= e^\lambda \prod_m (1 - e^{-\lambda m}) \text{tr}_{\mathcal{H}_K} e^{-\lambda L_0^K}. \tag{79}
\end{aligned}$$

This proves Eq. (69a).  $\square$

## 6 Discussion

The extension of the vanishing theorem to  $d > 1$  is straightforward. Write  $\hat{\mathcal{H}}$  such that

$$\hat{\mathcal{H}} = (\mathcal{F}(\alpha_{-m}^0, b_{-m}, c_{-m}; k^0) \otimes \mathcal{H}_s)^{L_0}, \tag{80}$$

where  $\mathcal{H}_s = \mathcal{F}(\alpha_{-m}^i; k^i) \otimes \mathcal{H}_K$ . The superscript  $i$  runs from 1 to  $d-1$ . Similarly, break up  $L_m$ . In particular,

$$L_0 = L_0^{(0)} + L_0^g + L_0^{(s)}, \quad (81a)$$

where

$$L_0^{(0)} + L_0^g = h^{(0)} + \sum_{m=1}^{\infty} m(N_m^0 + N_m^b + N_m^c) - 1, \quad (81b)$$

$$L_0^{(s)} = h^{(s)} + \sum_{i=1}^{d-1} \sum_{m=1}^{\infty} m N_m^i + L_0^K, \quad (81c)$$

$$h^{(0)} = -\alpha'(k^0)^2, \quad h^{(s)} = \sum_{i=1}^{d-1} \alpha'(k^i)^2. \quad (81d)$$

Just like  $\mathcal{H}_K$ , the spectrum of  $\mathcal{H}_s$  is bounded below and  $L_0^{(s)} \geq 0$  for  $(k^i)^2 \geq 0$ .<sup>8</sup> Thus, our derivation applies to  $d > 1$  essentially with no modification; simply make the following replacements:

$$\mathcal{H}_K \rightarrow \mathcal{H}_s, L_0^X \rightarrow L_0^{(0)}, L_0^K \rightarrow L_0^{(s)}, h^X \rightarrow h^{(0)}. \quad (82)$$

(However, use the only momentum independent piece of  $L_0^{(s)}$  in calculating the index and the signature.)

The standard proofs of the no-ghost theorem do not only show the theorem, but also show that the BRST cohomology is isomorphic to the light-cone or OCQ spectra. Since we do not have light-cone directions in general, we do not show this. In other words, we do not construct physical states explicitly.

Our proof does not apply at the exceptional value of momentum  $k^\mu = 0$  because the vanishing theorem fails (See footnote 6). Even in the flat  $d = 26$  case, the exceptional case needs a separate treatment [8, 9, 22]. For the flat case, the relative cohomology is nonzero at three ghost numbers and is represented by

$$\alpha_{-1}^\mu |0; k^\mu = 0\rangle, \quad b_{-1} |0; k^\mu = 0\rangle, \quad \text{and} \quad c_{-1} |0; k^\mu = 0\rangle. \quad (83)$$

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<sup>8</sup>In fact, FGZ have shown that an infinite sum of Verma modules with  $h \geq 0$  furnish a basis of the Fock space  $\mathcal{F}(\alpha_{-m}^i; k^i)$ . Thus, not only  $\mathcal{H}_K$ , but the whole  $\mathcal{H}_s$  must be written by Verma modules with  $h \geq 0$ .

Thus, there are negative norm states. However, the physical interpretation of these infrared states is unclear [8].

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## A The Künneth Formula

To simplify the notation, denote the complexes appeared in Section 4 as follows:

$$(\mathcal{F}(\alpha_{-m}^0, b_{-m}, c_{-m}; k^0), Q_0^{(FGZ)}) \rightarrow (\mathcal{F}, Q), \quad (84a)$$

$$(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0), d') \rightarrow (\mathcal{F}_1, d'), \quad (84b)$$

$$(\mathcal{F}(c_{-m}), d'') \rightarrow (\mathcal{F}_2, d''). \quad (84c)$$

Let  $\{\omega_i^{-b}\}$  and  $\{\eta_j^{n+b}\}$  be bases of  $H^{-b}(\mathcal{F}_1)$  and  $H^{n+b}(\mathcal{F}_2)$  respectively. Then,  $\phi^n = \omega_i^{-b} \eta_j^{n+b}$  is a closed state in  $\mathcal{F} (= \mathcal{F}_1 \otimes \mathcal{F}_2)$ . We show that this is not an exact state. If it were exact, it would be written as

$$\phi^n = \omega_i^{-b} \eta_j^{n+b} = Q(\alpha^{-b-1} \beta^{n+b} + \gamma^{-b} \delta^{n+b-1}) \quad (85)$$

for some  $\alpha^{-b-1}$ ,  $\beta^{n+b}$ ,  $\gamma^{-b}$ , and  $\delta^{n+b-1}$ . Executing the differential, we get

$$\begin{aligned} \omega_i^{-b} \eta_j^{n+b} &= (d' \alpha^{-b-1}) \beta^{n+b} + \alpha^{-b-1} (-)^{b+1} (d'' \beta^{n+b}) \\ &\quad + (d' \gamma^{-b}) \delta^{n+b-1} + \gamma^{-b} (-)^b (d'' \delta^{n+b-1}). \end{aligned} \quad (86)$$

Comparing the left-hand side with the right-hand side, we get  $\alpha^{-b-1} = \delta^{n+b-1} = 0$ ; thus,  $\phi^n = 0$  contradicting our assumption. Thus,  $\phi^n$  is an

element of  $H^n(\mathcal{F})$ . Conversely, any element of  $H^n(\mathcal{F})$  can be decomposed into a sum of a product of the elements of  $H^{-b}(\mathcal{F}_1)$  and  $H^{n+b}(\mathcal{F}_2)$ . Thus, we obtain

$$H^n(\mathcal{F}) = \bigoplus_{\substack{n=c-b \\ c, b \geq 0}} H^{-b}(\mathcal{F}_1) \otimes H^c(\mathcal{F}_2). \quad (87)$$

Here, the restriction of the values  $b$  and  $c$  comes from the fact  $\mathcal{F}_1^n = \mathcal{F}_2^{-n} = 0$  for  $n > 0$ .

This is the Künneth formula (for the “torsion-free” case [24].) Our discussion here is close to the one of ref. [25] for the de Rham cohomology.

## B Some Useful Commutators

In this appendix, we collect some useful commutators:

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}, & [L_m, b_n] &= (m - n)b_{m+n}, \\ [L_m, \alpha_n^\nu] &= -n\alpha_{m+n}^\nu, & [Q, \alpha_m^\nu] &= -\sum_n mc_n\alpha_{m-n}^\nu, \\ [L_m, c_n] &= (-2m - n)c_{m+n}, & \{Q, c_m\} &= -\sum_n nc_{-n}c_{m+n}, \\ [Q, L_m] &= 0, & [N^g, c_m] &= c_m, \\ \{Q, b_m\} &= L_m, & [N^g, Q] &= Q. \\ [N^g, b_m] &= -b_m, \\ [N^g, L_m] &= 0, \end{aligned}$$

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